Worcester Polytechnic Institute
Department of Mathematical Sciences

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General Comprehensive Examination

Linear Algebra

Print Name: ____________________________ Sign: ____________________________

Problem 1: Let $f_1, \ldots, f_n$ be real-valued continuous functions on the real interval $[a, b]$. For $1 \le i, j \le n$, let

$$g_{ij} = \int_a^b f_i(x) f_j(x)\, dx.$$

Prove: if the matrix $G = [g_{ij}]_{i,j}$ is singular, then the functions $f_1, \ldots, f_n$ are linearly dependent.

Problem 2: Prove that the eigenvalues of a self-adjoint (viz. Hermitian) matrix with complex entries are real.

Problem 3: Let $P_2(\mathbb{R})$ denote the vector space of all polynomials of degree at most two with real coefficients. Find $q(x) \in P_2(\mathbb{R})$ such that, for all $p(x) \in P_2(\mathbb{R})$,

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x) q(x)\, dx.$$

Problem 4: Let V be the real vector space spanned by

$$\{1,\ \cos x,\ \cos 2x,\ \ldots,\ \cos nx,\ \sin x,\ \sin 2x,\ \ldots,\ \sin nx\}.$$

with the scalar product

$$(f, g) = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

Let D be the differential operator, viz. $Df = f'$.

(i) Prove D is not self-adjoint.

(ii) Prove that the operator T defined by $Tf = f''$ is self-adjoint.

(iii) Prove that $-T$ is positive.

Problem 5: Let $M \in \mathsf{GL}_n(\mathbb{C})$ be a *nilpotent* matrix; that is, assume that there exists an integer k such that $M^k = 0$.

(i) Prove that every eigenvalue of M is 0;

(ii) Hence prove that $k \le n$;

(iii) Prove that $M + I$ is invertible, and find an expression for the inverse as a polynomial in I and M.